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The Rank 3 Permutation Representations of Finite Groups of Type G_2

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We say a group G^* is of type G_2 provided

$$G_2(q) \leqslant G^* \leqslant \mathrm{Aut}\ G_2(q).$$

where $q = p^f$, p is a prime.

In this paper, we prove

MAIN THEOREM. *There exists no faithful primitive permutation representation of rank* 3 *of* G^* *with* $p \geqslant 3$, *except when* $G^* = G_2(3)$ *and the degrees of such permutation representations are equal to* 351.

In Section 5 we will give two examples which are inequivalent primitive permutation representations of rank 3 of $G_2(3)$. The proof of the main theorem is continued in the following sections.

1. PRELIMINARIES

By definition $G_2(q) \trianglelefteq G^* \leqslant \mathrm{Aut}\ G_2(q)$. For brevity, we often denote $G_2(q)$ by G. We know that

$$\mathrm{Aut}\ G = \langle G, \text{the field automorphisms of } G, \text{the Graph automorphisms of } G\rangle.$$

The graph automorphism of $G_2(q)$ is the identity except for $G_2(3^f)$, where unique graph automorphism η is of order $2f$ and η^2 is a field automorphism. The only diagonal automorphism of $G_2(q)$ is the identity.

Let $G^{\#} = \langle G$, the field automorphisms of $G\rangle$ and $G^+ = G^{\#} \cap G^*$. Then $|G^* : G^+| \mid 2$ and $|G^* : G^+| = 1$ except possibly when $q = 3^f$. We have also

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$G \trianglelefteq G^+ \trianglelefteq G^* \leqslant \mathrm{Aut}\, G$, $|G^+ : G| \,|\, f$ and $|G^* : G| \,|\, 2f$. Furthermore G^*/G is cyclic.

Let B^+ and B be the Borel subgroups of G^+ and G, respectively, and let U be the p-Sylow subgroup of G, then B^+ and B are the normalizers of U in G^+ and G, respectively, and $B = B^+ \cap G$. Moreover, $G^+ = G \cdot B^+$.

Now G and G^+ have the common Weyl group W. Let α_1, α_2 be the simple roots of G_2, s_1, s_2 are the reflections corresponding to α_1, α_2, respectively. W is of order 12. We can regard the elements of W as ones of G.

Let $W_1 = \langle 1, s_2 \rangle$ and $W_2 = \langle 1, s_1 \rangle$, then W has four $W_i - W_i$ double cosets and from Mackey's theorem

$$(1_{W_i}^W, 1_{W_i}^W) = 4 \qquad (i = 1, 2).$$

Let $G_1^+ = \langle B^+, W_1 \rangle = \langle B^+, X_{-\alpha_2} \rangle$ and $G_2^+ = \langle B^+, W_2 \rangle = \langle B^+, X_{-\alpha_1} \rangle$, where $X_{-\alpha_1}$, $X_{-\alpha_2}$ are root subgroups of G^+. G_1^+, G_2^+ are the maximal parabolic subgroups of G^+, and we have

$$(1_{G_i^+}^{G^+}, 1_{G_i^+}^{G^+}) = (1_{W_i}^W, 1_{W_i}^W) = 4 \qquad (i = 1, 2).$$

Furthermore, $1_{B^+}^{G^+}|_G = 1_B^G$ and restricting to G, the irreducible constituents of $1_{B^+}^{G^+}$ are still irreducible.

Now suppose that φ^* is a faithful primitive permutation representation of G^* of rank 3, K^* is the stable subgroup of G^* leaving one letter fixed. Then $\varphi^* = 1_{K^*}^{G^*}$.

Since φ^* is primitive and $G \trianglelefteq G^*$, G is transitive and $G^* = G \cdot K^*$. Let $K^+ = G^+ \cap K^*$, $K = G \cap K^* = G \cap K^+$, $\varphi^+ = \varphi^*|_{G^+}$ and $\varphi = \varphi^*|_G = \varphi^+|_G$, then $\varphi^+ = 1_{K^+}^{G^+}$ and $\varphi = 1_K^G$.

We also denote by φ^*, φ^+, φ the characters of φ^*, φ^+, and φ, respectively. We consider three cases,

(1) $K^+B^+ = G^+$, namely, $(1_{K^+}^{G^+}, 1_{B^+}^{G^+}) = 1$,

(2) $(\varphi^+(1), p) = 1$,

(3) $(1_{K^+}^{G^+}, 1_{B^+}^{G^+}) \geqslant 2$ and $p \,|\, \varphi^+(1)$.

For case (1), K^+ is a flag-transitive subgroup of G^+ and from [14, Theorem A], we have

LEMMA 1.1. *Case* (1) *does not occur.*

In Section 2 we prove that there exists no subgroup K^* of G^* which satisfies (2) with $1_{K^*}^{G^*}$ primitive and of rank 3. We discuss case (3) in Section 3–4.

2. The Proof in Case 2

Since $\varphi^+(1)|K^+|=|G^+|$ and $(\varphi^+(1),p)=1$, $|K^+|$ and $|G^+|$ are divisible by the same power of p. The same fact holds for K and G. Up to conjugacy, K contains the p-Sylow subgroup U of G and K^+ contains a p-Sylow subgroup of G^+ which contains U. From Tits lemma [14, (1.6)], $K^+ \leqslant G_i^+$ ($i=1$ or 2) or G is contained in a maximal subgroup L of G^+ which also contains K^+. For the latter case, we have $G^+=G\cdot K^+ \leqslant L$, this is impossible. Thus $K^+ \leqslant G_i^+$.

Hence $(1_{K^+}^{G^+}, 1_{K^+}^{G^+}) \geqslant (1_{G_i^+}^{G^+}, 1_{G_i^+}^{G^+})=4$. But $1_{K^*}^{G^*}$ is of rank 3 and $1_{K^+}^{G^+}=1_{K^+}^{G^*}|_{G^+}$, we have

$$G^+ \lneqq G^* \qquad \text{and} \qquad |G^*:G^+|=2.$$

G^* has one graph automorphism (At this moment, of course $q=3^f$).

Now we prove a general lemma.

Lemma 2.1. *Let G be a finite Chevalley group defined over F_q, $q=p^f$, with $Z(G)=1$. Let $G^{\#}=\langle G$, the field automorphisms of G, the diagonal automorphisms of $G\rangle$. Given G^*, $G \trianglelefteq G^* \leqslant \operatorname{Aut} G$, let $G^+=G^*\cap G^{\#}$, then $G \trianglelefteq G^+ \trianglelefteq G^*$. Assume $G^+ \lneqq G^*$ and $1_{K^*}^{G^*}$ is primitive. Let $K^+=G^+\cap K^*$, $K=G\cap K^*=G\cap K^+$. And let U be the p-Sylow subgroup of G and $G_I^+=\langle B^+, \bigcup_{\alpha\notin I} X_{-\alpha}\rangle$ be a parabolic subgroup of G^+, where I is a subset of simple roots of G and $(X_{-\alpha})$'s are root subgroups. If*

$$U \leqslant K \leqslant K^+ \leqslant G_I^+,$$

then there exists a graph automorphism $\eta \neq 1$ of G such that

$$K^+ \leqslant G_I^+ \cap G_{I\eta}^+.$$

Proof. (i) From an argument of Steinberg [16, proof of Theorem 30] we have the following assertion: If σ is an automorphism of G such that $U^\sigma=U$, then we can multiply by an inner-automorphism corresponding to an element $u\in U$ so that $U^{\sigma u}=U$ and σu permutes the root subgroups of G.

(ii) Since $G \trianglelefteq G^*$ and $1_{K^*}^{G^*}$ is primitive, G is transitive and $G^*=G\cdot K^*$. We have $K^+ \lneqq K^*$, as $G^+\neq G^*$. Thus $K^*\backslash G^+ \neq \varnothing$. And from $K^+ \lhd K^*$, we can find an element $z_1\in G^*\backslash G^+$ such that $(K^+)^{z_1}=K^+$. Now $K=G\cap K^+$ and $G \lhd G^*$, we have $K^{z_1}=(G\cap K^+)^{z_1}=G\cap K^+=K$. Since U is the p-Sylow subgroup of K, there exists $k\in K\subseteq K^+$ such that $U^{z_1k}=U$. Also from (i), we can find $u\in U$ which satisfies $U^{z_1ku}=U$ and meets the requirement in (i). Clearly, $ku\in K^+$, thus $z=z_1ku\notin G^+$ and $(K^+)^{z_1ku}=K^+$.

(iii) Since $z \in G^*$ and $z \notin G^+ = G^* \cap G^{\#}$, $z \notin G^{\#}$. Then there exists a graph automorphism $\eta \neq 1$ of G and an element $g \in G^{\#}$ such that $z = \eta g$.

Both the actions of z and η permute the root subgroups of G and leave U fixed, so does g. But $g \in G^{\#}$, the permutation is the identity. Thus the two permutations corresponding to z and η are the same. And $(G^+)^z = G^+$ since $G^+ \lhd G^*$. Thus z leaves $B^+ = N_{G^+}(U)$ fixed.

Now $G_I^+ = \langle B^+, \bigcup_{\alpha \notin I} X_{-\alpha} \rangle$, thus $(G_I^+)^z = \langle (B^+)^z, \bigcup_{\alpha \notin I} (X_{-\alpha})^z \rangle = \langle B^+, \bigcup_{\alpha \notin I^\eta} X_{-\alpha} \rangle = G_{I^\eta}^+$. From the assumption of the lemma that $K^+ \leqslant G_I^+$, we have

$$K^+ = (K^+)^z \leqslant (G_I^+)^z = G_{I^\eta}^+.$$

Hence

$$K^+ \leqslant G_I^+ \cap G_{I^\eta}^+.$$

The proof is complete.

THEOREM 2.3. *There exists no subgroups K^* of G^* such that $\varphi^* = 1_{K^*}^{G^*}$ is of rank* 3 *and* $(\varphi^*(1), p) = 1$, *where G^* is as in Section* 1.

Proof. Suppose that a subgroup K^* of G^* satisfies

$$1_{K^*}^{G^*} = \varphi^* = 1 + \chi^* + \zeta^*,$$

where 1 means the identity representation χ^*, ζ^* are irreducible and $(\varphi^*(1), p) = 1$. Restricting φ^*, χ^*, and ζ^* to G^+, we get φ^+, χ^+, and ζ^+. Since $G^+ \lhd G^*$ and $|G^* : G^+| \mid 2$, from Clifford's theorem, [3, (53.17)] and [10, (9.12)], both χ^+ and ζ^+ are multiplicity free and have at most two irreducible constituents which are conjugates to each other. Then $\chi^+ + \zeta^+$ has at most four irreducible constituents and the multiplicity of each irreducible constituent is at most 2. Thus $1_{K^*}^{G^*}|_{G^+} = 1_{K^+}^{G^+}$ is of rank at most 9.

Now $\varphi^*(1) = \varphi^+(1)$, $(\varphi^+(1), p) = 1$ and from the results in the beginning of this section we get $U \leqslant K^+ \leqslant G_i^+$ ($i = 1$ or 2). And then all the conditions in Lemma 2.1 hold. (At this moment $G^+ \lneqq G^*$ and $q = 3^f$.)

For the unique nontrivial graph automorphism η of $G_2(3^f)$, we have $\alpha_1^\eta = \alpha_2$, $\alpha_2^\eta = \alpha_1$, thus $K^+ \leqslant G_{\{\alpha_1\}}^+ \cap G_{\{\alpha_2\}}^+ = G_1^+ \cap G_2^+$. Notice that $G_1^+ = B^+ \cup B^+ s_2 B^+$, $G_2^+ = B^+ \cup B^+ s_1 B^+$, thus $G_1^+ \cap G_2^+ = B^+$ and then $K^+ \leqslant B^+$. We get

$$(1_{K^+}^{G^+}, 1_{K^+}^{G^+}) \geqslant (1_{B^+}^{G^+}, 1_{B^+}^{G^+}) = |W| = 12.$$

Hence $1_{K^+}^{G^+}$ is of rank more than 12. This contradicts the fact that $1_{K^+}^{G^+}$ is of rank at most 9 and the proof is complete.

The main task of this paper is to deal with case (3).

3. Some Consequences of the Representation Theory of $G_2(q)$

As above, $\varphi^+ = 1 + \chi^+ + \zeta^+$, both χ^+ and ζ^+ are multiplicity free and contain at most two irreducible constituents which are conjugate to each other under G^*. By the Frattini argument, $G^* = G^+ N(B^+)$. Then 1_{B+}^{G+} is invariant under G^*. Since $(1_{K+}^{G+}, 1_{B+}^{G+}) \geqslant 2$, $\varphi^+ = 1_{K+}^{G+}$ and 1_{B+}^{G+} contain an irreducible constituent besides 1-representation in common. Assume that χ^+ contains this constituent. Since 1_{B+}^{G+} is invariant under G^*, the conjugate constituent under G^* is still in 1_{B+}^{G+}. Thus all of χ^+ is in 1_{B+}^{G+} as χ^+ is multiplicity free. Now we restrict $\varphi^+ = 1_{K+}^{G+} = 1 + \chi^+ + \zeta^+$ to $G = G_2(q)$ and let

$$\varphi = \varphi^+|_G, \qquad \chi = \chi^+|_G, \qquad \text{and} \qquad \zeta = \zeta^+|_G.$$

We have $\varphi = 1 + \chi + \eta$ and $p \mid \varphi(1)$. Notice that $\varphi = 1_K^{G_2}$, we have $\varphi(1) = |G_2|/|K|$ and $\varphi(1) \,\|\, |G_2|$.

From now on we assume that $\varphi(1)$ satisfies the following conditions:

$$p \mid \varphi(1), \tag{1}$$

$$\varphi(1) \,\|\, |G_2|. \tag{2}$$

Restrict 1_{B+}^{G+} and its irreducible constituents to G. Since $1_{B+}^{G+}|_G = 1_B^G$ and $(1_{B+}^{G+}, 1_{B+}^{G+}) = |W| = (1_B^G, 1_B^G)$, we get a one-to-one correspondence between the irreducible constituents of 1_{B+}^{G+} and 1_B^G and the correspondence preserves the equivalence between constituents. Because of this property of χ^+, all of χ is in 1_B^G and it is multiplicity free, it contains at most two irreducible constituents and they are conjugate to each other under G^*.

From Chang and Ree [1] and Enomoto [5], $1_B^{G_2}$ contains six inequivalent irreducible constituents. We denote their characters by χ_1 (the principal character), χ_2, χ_3, χ_4, χ_5, and χ_6. Their degrees are the following:

$$\chi_1(1) = 1,$$

$$\chi_2(1) = q^6,$$

$$\chi_3(1) = \chi_4(1) = \tfrac{1}{3} q(q^4 + q^2 + 1),$$

$$\chi_5(1) = \tfrac{1}{2} q(q+1)^2 (q^2 - q + 1),$$

$$\chi_6(1) = \tfrac{1}{6} q(q+1)^2 (q^2 + q + 1).$$

Furthermore, χ_1, χ_2, χ_5, and χ_6 are self-conjugate and χ_3, χ_4 are conjugate to each other under G^*. Thus the only possibilities for $\chi(1)$ are $\chi_2(1)$, $\chi_3(1)$, $\chi_3(1) + \chi_4(1)$, $\chi_5(1)$, $\chi_6(1)$. It is easy to see that $p \mid \chi(1)$ except for $q = 3$.

Now look at $\zeta = \zeta^*|_G$. From Clifford theorem and [10, (9.12)] we can assume that ζ splits into k irreducible constituents which are conjugate to each other under G^* and ζ is multiplicity free. Since $|G^* : G_2| \mid 2f$, where

$q=p^f$, from [3, (53.17)], $k|2f$. Denote by ξ the character of one of these constituents of ζ, then $\zeta(1)=k\xi(1)$.

In this section we will determine which combinations of $\chi(1)$'s, $\zeta(1)$'s and k's are such that relations (1) and (2) hold.

We now consider four cases:

(i) $q=3$,

(ii) $q=3^2$, $k\neq 1$, and $\chi=\chi_3, \chi_4, \chi_3+\chi_4$, and χ_6,

(iii) $q\neq 3$, $\zeta(1)=k\xi(1)$, and $k=1$,

(iv) The remaining cases.

Case (i). $q=3$.

In this case $f=1$, $k|2$. Then k has at most two possibilities, 1, 2.

Now $\chi_2(1)=729$, $\chi_3(1)=\chi_4(1)=91$, $\chi_3(1)+\chi_4(1)=182$, $\chi_5(1)=104$, $\chi_6(1)=168$. And from Enomoto [5], we find the degrees of the remaining irreducible representations of $G_2(3)$ as follows: $\theta_6(1)=91$, $\theta_7(1)=819$, $\theta_8(1)=\theta_9(1)=273$, $\theta_{10}(1)=14$, $\theta_{11}(1)=78$, $\theta_{12}(1)=64$, $\chi_1(K)(1)=\chi_3(K)(1)=\chi_{12}(K,l)(1)=364$, $\chi_2(K)(1)=\chi_4(K)(1)=3\times 364$, $\chi_5(K)(1)=\chi_7(K)(1)=182$, $\chi_6(K)(1)=\chi_8(K)(1)=546$, $\chi_9(K,l)(1)=4\times 364$, $\chi_{10}(K)(1)=\chi_{11}(K)(1)=728$. $\chi_{13}(K)(1)=448$, $\chi_{14}(K)(1)=832$, where θ_i, $\chi_j(K)$, $\chi_t(K,l)$ is the notation for the irreducible representaztions of $G_2(q)$ as in [5].

It is well known that $|G_2(q)|=q^6(q^2-1)(q^6-1)$. In the case $q=3$, $|G|=2^6\cdot 3^6\cdot 13\cdot 7$.

By simple computation we get all the combinations which satisfy (1), (2), and the properties mentioned above. They are

$$1+91+64=156,$$

$$1+2\cdot 91+2\cdot 273=729=q^6,$$

$$1+104+168=273,$$

$$1+168+182=351,$$

$$1+729+728=2q^6.$$

$$1+2\cdot 91+168=351,$$

$$1+2\cdot 91+546=729=q^6,$$

$$1+104+273=378,$$

$$1+729+2\cdot 364=2\cdot 729=2q^6,$$

In Cases (ii), (iii), and (iv), $q=p^f\neq 3$, so $p|\chi(1)$. And condition (1) requires $p|\varphi(1)$, then $p|1+k\xi(1)$ and $(p,\xi(1))=1$. From [1, 5] the possibilities of $\xi(1)$ are

$\xi_1(1) = q^4 + q^2 + 1,$ $\xi_5(1) = (q^2 + q + 1)(q^3 + 1),$

$\xi_2(1) = (q^2 - q + 1)(q^3 - 1),$ $\xi_6(1) = (q + 1)(q^2 + q + 1)(q^3 + 1),$

$\xi_3(1) = q^6 - 1,$ $\xi_7(1) = (q - 1)(q^2 - q + 1)(q^3 - 1),$

$\xi_4(1) = (q - 1)(q^2 - 1)(q^3 + 1),$ $\xi_8(1) = (q + 1)(q^2 - 1)(q^3 - 1),$

$\xi_9(1) = q^3 \pm 1$ with $q = \pm 1 \pmod 3$.

Case (ii). $q = 3^2$, $k \neq 1$ and $\chi = \chi_3, \chi_4, \chi_3 + \chi_4$, and χ_6. In this case, $f = 2$, $k | 4$, so $k = 2$ or 4. Since $q \equiv 0 \pmod 3$, ξ_9 is excluded. $\chi(1)$ and $\xi_j(1)$ are

$\chi_3(1) = \chi_4(1) = 3 \cdot 6643,$ $\chi_3(1) + \chi_4(1) = 6 \cdot 6643,$ $\chi_6(1) = 13650,$

$\xi_1(1) = 6643,$ $\xi_5(1) = 66430,$

$\xi_2(1) = 53144,$ $\xi_6(1) = 664300,$

$\xi_3(1) = 531440,$ $\xi_7(1) = 425152,$

$\xi_4(1) = 467200,$ $\xi_8(1) = 582400.$

By simple computation we find that none of $\varphi(1) = 1 + \chi(1) + k\xi(1)$ which are in case (ii) satisfies conditions (1) and (2).

Case (iii). $q \neq 3$, $\zeta(1) = k\xi(1)$, $k = 1$.

$$\varphi(1) = 1 + \chi(1) + \xi_j(1), \qquad \chi(1) = \chi_i(1) \quad \text{or} \quad 2\chi_3(1) \quad (i = 2, 3, 5, 6).$$

Since $p | 1 + \xi_j(1)$, $\xi_j(1)$ are just

$$\xi_2(1) = (q^2 - q + 1)(q^3 - 1),$$
$$\xi_3(1) = q^6 - 1,$$
$$\xi_9(1) = q^3 - 1 \qquad \text{with} \quad q \equiv -1 \pmod 3).$$

In the proof of Cases (iii) and (iv) we will use the resultant. Given

$$f(x) = a_0\chi^n + a_1\chi^{n-1} + \cdots a_n,$$
$$g(x) = b_0\chi^m + b_1\chi^{m-1} + \cdots + b_m,$$

let $t_1, \ldots, t_n$ be the roots of $f(x)$. The resultant

$$R(f, g) = a_0^m \prod_{i=1} g(t_i).$$

It has the property that if q, a_0, $a_1,\ldots,a_n$; b_0, $b_1,\ldots,b_m$ are integers, then $R(f, g)$ is an integer and

$$(f(q), g(q)) \mid R(f, g).$$

Now we compute $\varphi(1) = 1 + \chi(1) + \xi_j(1)$ and examine which of $\varphi(1)$ satisfy condition (2).

We describe the method by the following example,

$$\begin{aligned}\varphi(1) &= 1 + \chi_6(1) + \xi_2(1)\\ &= 1 + \tfrac{1}{6}q(q+1)^2\,(q^2+q+1) + (q^2-q+1)(q^3-1).\end{aligned}$$

When $q \equiv 0, -1 \pmod 3$, $\varphi(1)$ is prime to q^2+q+1. When $q \equiv 1 \pmod 3$, $3 \mid q^2+q+1$ and $\varphi(1)$ is prime to $\frac{1}{3}(q^2+q+1)$. Also $\varphi(1)$ is equal to $\frac{1}{6}q(q^2-q+1)(7q^2+4q+7)$. Since $\varphi(1)$ divides $|G_2(q)| = q^6(q^2-1)(q^6-1) = q^6(q^2-1)^2\,(q^2+q+1)(q^2-q+1)$, we have

$$7q^2+4q+7 \mid 18 \cdot q^5(q^2-1)^2.$$

If $q = 7^f$, then 7 divides the left side and 7^2 does not. If $q \neq 7^f$, $(q, 7) = 1$. In any case we have

$$7q^2+4q+7 \mid 18 \cdot 7(q^2-1)^2. \tag{3}$$

For brevity, we denote $R((x^2-1)^2, 7x^2+4x+7)$ by $R(\text{right, left})$. We have

$$R(\text{right, left}) = 18^2 \cdot 10^2.$$

Since

$$\begin{aligned}&7q^2+4q+7 = (7q^2+4q+7,\ 18 \cdot 7(q^2-1)^2),\\ &7q^2+4q+7 \mid 18 \cdot 7((q^2-1)^2, 7q^2+4q+7)\end{aligned}$$

and

$$7q^2+4q+7 \mid 18 \cdot 7\,R(\text{right, left}).$$

Thus

$$7q^2+4q+7 \mid 18 \cdot 7 \cdot 18^2 \cdot 10^2. \tag{4}$$

Computing we find that q does not satisfy this relation except for $q = 7$.

For $q = 7$, the relation, $\varphi(1) \,\big|\, |G_2(7)|$, holds.

By the same method as in this example we examine all $\varphi(1) = 1 + \chi(1) + \xi_j(1)$. For brevity we write only the reduced divisibility relations like (3), (4), and q for which $\varphi(1)$ satisfies condition (2).

(1) $1 + \chi(2) + \xi_2(1) = 1 + q^6 + (q^3 - 1)(q^2 - q + 1).$
$q^5 + q^4 - q^3 + q^2 - q + 1 \mid (q^2 - 1)^2,$
$q^5 + q^4 - q^3 + q^2 - q + 1 \mid 2^6,$
none.

(2) $1 + \chi_2(1) + \xi_3(1) = 1 + q^6 + (q^6 - 1) = 2q^6,$
$2q^6 \,\big|\, |G_2(q)|$ for all $q = p^f,\ p \geqslant 3.$

(3) $1 + \chi_2(1) + \xi_9(1) = 1 + q^6 + (q^3 - 1)$
$= q^3(q^3 + 1) \quad (q \equiv -1, \bmod 3)$
$q^3(q^3 + 1) \,\big|\, |G_2(q)|$ for all q as above.

(4) $1 + \chi_3(1) + \xi_2(1) = 1 + \frac{1}{3}q(q^4 + q^2 + 1) + (q^2 - q + 1)(q^3 - 1),$
$4q^4 - 3q^3 + 4q^2 - 3q + 4 \mid 9(q^2 - 1)^2,$
$4q^4 - 3q^3 + 4q^2 - 3q + 4 \mid 2^4 \cdot 3^8,$
none.

(5) $1 + \chi_3(1) + \xi_3(1) = 1 + \frac{1}{3}q(q^4 + q^2 + 1) + (q^6 - 1)$
$3q^4 - 2q^3 + 2q^2 - q + 1 \mid 9(q^2 - 1)(q - 1),$
$3q^4 - 2q^3 + 2q^2 - q + 1 \mid 3^6,$
none.

(6) $1 + \chi_3(1) + \xi_9(1) = 1 + \frac{1}{3}q(q^4 + q^2 + 1) + q^3 - 1$
$(q \equiv -1, \bmod 3)$
$q^4 + 4q^2 + 1 \mid 9(q^2 - 1)^2,$
$q^4 + 4q^2 + 1 \mid 2^4 \cdot 3^6,$
none.

(7) $1 + \chi_5(1) + \xi_2(1)$
$= 1 + \frac{1}{2}q(q + 1)^2\,(q^2 - q + 1) + (q^2 - q + 1)(q^3 - 1)$
$3q^2 - 4q + 3 \mid 6(q^2 - 1)^2,$
$3q^2 - 4q + 3 \mid 2^5 \cdot 3 \cdot 5^2,$
none.

(8) $1 + \chi_5(1) + \xi_3(1) = 1 + \frac{1}{2}q(q + 1)^2\,(q^2 - q + 1) + q^6 - 1$
$2q^3 - q^2 + 1 \mid 4(q - 1)^2,$
$2q^3 - q^2 + 1 \mid 2^4,$
none.

(9) $1+\chi_5(1)+\xi_9(1)=1+\frac{1}{2}q(q+1)^2(q^2-q+1)+q^3-1$

$(q\equiv -1, \bmod 3)$

$q^2+1 \mid 4(q-1)^2,$

$q^2+1 \mid 2^4,$

none.

(10) $1+\chi_6(1)+\xi_2(1)$

$=1+\frac{1}{6}q(q+1)^2(q^2+q+1)+(q^2-q+1)(q^3-1)$

$7q^2+4q+7 \mid 18\cdot 7(q^2-1)^2,$

$7q^2+4q+7 \mid 2^5\cdot 3^6\cdot 5^2\cdot 7,$

$q=7.$

(11) $1+\chi_6(1)+\xi_3(1)=1+\frac{1}{6}q(q+1)^2(q^2+q+1)+q^6-1$

$6q^3+7q^2+4q+1 \mid 36(q-1)^2,$

$6q^3+7q^2+4q+1 \mid 2^4\cdot 3^6,$

none.

(12) $1+\chi_6(1)+\xi_9(1)=1+\frac{1}{6}q(q+1)^2(q^2+q+1)+q^3-1$

$(q\equiv -1, \bmod 3)$

$q^4+3q^3+10q^2+3q+1 \mid 36(q-1)^2(q^2-q+1),$

$q^4+3q^3+10q^2+3q+1 \mid 2^8\cdot 3^8,$

none.

(13) $1+\chi_3(1)+\chi_4(1)+\xi_2(1)$

$=1+\frac{2}{3}q(q^4+q^2+1)+(q^2-q+1)(q^3-1)$

$5q^4-3q^3+5q^2-3q+5 \mid 45(q^2-1)^2,$

$5q^4-3q^3+5q^2-3q+5 \mid 5\cdot 3^8\cdot 7^2,$

none.

(14) $1+\chi_3(1)+\chi_4(1)+\xi_3(1)=1+\frac{2}{3}q(q^4+q^2+1)+(q^6-1)$

$3q^5+2q^4+2q^2+2 \mid 3^2(q^2-1)^2,$

$3q^5+2q^4+2q^2+2 \mid 3^8,$

none.

(15) $1+\chi_3(1)+\chi_4(1)+\xi_9(1)$

$=1+\frac{2}{3}q(q^4+q^2+1)+(q^3-1)\ \ (q\equiv -1, \bmod 3)$

$2q^4+5q^2+2 \mid 9(q^2-1)^2,$

$2q^4+5q^2+2 \mid 3^{10},$

none.

Case (iv). The remaining cases. $\varphi(1) = 1 + \chi(1) + k\xi(1)$. Now $q \neq 3$, $k \neq 1$ and when $q = 3^f$ and $\chi = \chi_3, \chi_4, \chi_3 + \chi_4$, and χ_6, we have $f > 2$. In this case $\xi(1) = \xi_j(1)$, $j = 1, 2, \ldots, 9$ and $\xi_j(1)$ are of form $qf(q) \pm 1$, where $f(q)$ is a polynomial of q with integer coefficients.

LEMMA 3.1. *In Case* (iv), *if* $\varphi(1)$ *satisfies condition* (1), *then* $p \mid k + 1$ *or* $k - 1$, *depending on* $\xi(1) = qf(q) + 1$ *or* $qf(q) - 1$. *And if* $p^r \| k + 1$ (*or* $k - 1$), *then* $p^r \| \varphi(1)$.

Proof. $\xi_j(1)$ is of form $qf(q) \pm 1$, $f(q)$ is an integer. Since $q = p^f \neq 3$, $p \mid \chi(1)$. From $p \mid \varphi(1)$, we have $p \mid k + 1$ (or $k - 1$).

For $q \neq 3^f$ or $\chi = \chi_2, \chi_5$, we have $q \mid \chi(1)$. Since $k \mid 2f$ and $q > 3$, it is easy to prove that $q = p^f > 2f + 1 \geqslant k \pm 1$. Thus $r < f$ and $p^r \| \varphi(1)$ since $\varphi(1) = \chi(1) + kqf(q) \pm (k \pm 1)$.

For $q = 3^f$ and $\chi(1) = \chi_3(1)$, $\chi_3(1) + \chi_4(1)$, $\chi_6(1)$, we are not in Cases (i) or (ii), so $f > 2$. It is easy to prove that $\frac{1}{3}q = 3^{f-1} > 2f + 1 \geqslant k + 1$. Thus $r < f - 1$. By looking at the expressions of $\chi(1)$, we have $3^{f-1} \mid \chi(1)$ and $3^r \| \varphi(1)$.

LEMMA 3.2. *Let* $s(p, t)$ *be the function with integer values defined by*

$$p^{s(p,t)} \leqslant 2t + 1 < p^{s(p,t)+1},$$

where $p > 1$ *and* t *are positive integers.*

If

$$\frac{1}{p^{s(p,t)}} (p^f)^l > a \prod_{i=1}^{m} (a_i \cdot 2f + b_i), \qquad (*)$$

where $a > 0$, $p^{l-1} \geqslant (1 + 1/f)^m$, $l \geqslant 2$, $a_i > 0$, $b_i \geqslant 0$ $(i = 1, 2, \ldots, m)$ *and* $c \geqslant p$, $g \geqslant f$, *then* $(*)$ *still holds when we replace* p, f *by* c *and* g *in* $(*)$. *Furthermore, for any* $k \leqslant 2g$ *and* $c^r \| k + 1$ *or* $k - 1$, *we have*

$$\frac{1}{c^r} (c^g)^l > a \prod_{i=1}^{m} (a_i k + b_i). \qquad (**)$$

Proof. In order to prove $(*)$ we consider two cases.

(1) $c \geqslant p$, $g = f$. Clearly $s(c, f) \leqslant s(p, f)$. And $lf \geqslant 2f \geqslant p^{s(p,f)} - 1 \geqslant s(p, f)$.

Thus

$$\left(\frac{c}{p}\right)^{lf} \geqslant \left(\frac{c}{p}\right)^{s(p,f)} \geqslant \frac{c^{s(c,f)}}{p^{s(p,f)}}.$$

Hence

$$\frac{1}{c^{s(c,g)}}(c^f)^l \geqslant \frac{1}{p^{s(p,f)}}(p^f)^l,$$

and $(*)$ holds.

(2) $c=p$, $g=f+1$. We have

$$(a_i \cdot 2(f+1)+b_i) = \frac{f+1}{f} \cdot 2a_i f + b_i \leqslant \left(1+\frac{1}{f}\right)(a_i \cdot 2f + b_i)$$

and

$$a \prod_{i=1}^{m} (a_i \cdot 2(f+1)+b_i) \leqslant \left(1+\frac{1}{f}\right)^m \cdot a \prod_{i=1}^{m} (a_i \cdot 2f + b_i).$$

Since

$$p^{s(p,f)+2} = p \cdot p^{s(p,f)+1} > p \cdot (2f+1) \geqslant 2f+3 \geqslant p^{s(p,f+1)},$$

$s(p,f+1) \leqslant s(p,f)+1$. Then

$$\begin{aligned}\frac{1}{p^{s(p,f+1)}}(p^{f+1})^l &\geqslant \frac{1}{p^{s(p,f)+1}}(p^f)^l \cdot p^l \\ &= \frac{(p^f)^l}{p^{s(p,f)}} p^{l-1} > \left(1+\frac{1}{f}\right)^m a \prod_{i=1}^{m} (a_i \cdot 2f + b_i) \\ &\geqslant a \prod_{i=1}^{m} (a_i \cdot 2(f+1)+b_i).\end{aligned}$$

Hence $(*)$ holds.

For $k \leqslant 2g$, $c^r \| k+1$ or $k-1$, we have

$$c^r \leqslant k+1 \text{ (or } k-1) \leqslant 2g+1.$$

Thus $r \leqslant s(c,g)$ and $(1/c^r)(c^g)^l \geqslant (1/c^{s(c,g)})(c^g)^l$. Hence $(**)$ holds.

Now we prove case by case that none of $\varphi(1) = 1+\chi(1)+k\xi(1)$ in Case (iv) satisfies both conditions (1) and (2).

We describe the method by the following examples:

EXAMPLE 1. $1+\chi_2(1) = k\xi_1(1) = 1+q^6+k(q^4+q^2+1)$. Where $q=p^f$, $q \neq 3$, $k \mid 2f$. We know $|G_2(q)| = q^6(q^2-1)(q^6-1)$. Assume $p^r \| k+1$. Suppose

$$1+q^6+k(q^4+q^2+1) \mid q^6(q^2-1)(q^6-1),$$

then

$$1 + q^6 + k(q^4 + q^2 + 1) \mid p^r(q^2 - 1)(q^6 - 1). \tag{5}$$

We denote $R((x^2 - 1)(x^6 - 1), 1 + x^6 + k(x^4 + x^2 + 1))$ by $R(\text{right, left})$, then

$$R(\text{right, left}) = 2^4(3k + 2)^4.$$

Since $1+q^6+k(q^4 + q^2 + 1) = (p^r(q^2 - 1)(q^6 - 1), 1+q^6 + k(q^4 + q^2 + 1))$,

$$\frac{1}{p^r}(1 + q^6 + k(q^4 + q^2 +)) \mid 2^4(3k + 2)^4. \tag{6}$$

Let $p = 3$, $q = 3^3$, then $2f = 6$, $s(3,f) = 1$. For this set of values we have

$$\frac{1}{p^{s(p,f)}} q^6 > 2^4(3 \cdot 2f + 2)^4. \tag{7}$$

From (3, 2), for $p \geqslant 3, f \geqslant 3$, and $k \mid 2f$, we get the relation

$$\frac{1}{p^r} q^6 > 2^4(3k + 2)^4.$$

Then in (6), the left side is greater than the right side and (6) does not hold.

The remaining cases for which $q = p^f$ such that there exists k with $k \mid 2f$ and $p \mid k + 1$ are $q = 3^2$, $k = 2$; $q = 5^2$, $k = 4$.

Direct computation shows that neither of them satisfies condition (2).

EXAMPLE 2. $1 + \chi_6(1) + k\xi_2(1) = 1 + \frac{1}{6}q(q + 1)^2 (q^2 + q + 1) + k(q^2 - q + 1)(q^3 - 1)$. Suppose $1 + \frac{1}{6}q(q + 1)^2 (q^2 + q + 1) + k(q^2 - q + 1) \times (q^3 - 1) \mid q^6(q^2 - 1)(q^6 - 1)$, then $1 + \frac{1}{6}q(q + 1)^2 (q^2 + q + 1) + k(q^2 - q + 1)(q^3 - 1) \mid p^r(q^2 - 1)(q^6 - 1)$. When $q \not\equiv 1 \pmod 3$, $\varphi(1)$ is prime to $q^2 + q + 1$. When $q \equiv 1 \pmod 3$, $\varphi(1)$ is prime to $\frac{1}{3}(q^2 + q + 1)$. In any case we have

$$1 + \tfrac{1}{6}q(q + 1)^2(q^2 + q + 1) + k(q^2-q + 1)(q^3-1) \mid 3p^r(q^2-1)(q^2-q + 1).$$

It is easy to compute that

$$1 + \frac{1}{6}q(q + 1)^2 (q^2 + q + 1) + k(q^2 - q + 1)(q^3 - 1)$$

$$= \frac{q}{6}(q^2 - q + 1)(q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1)).$$

Thus we obtain

$$q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1) \mid 18 \cdot p^r(q^2 - 1)^2. \tag{8}$$

We have

$$\begin{aligned} R(\text{right, left}) &= R((x^2 - 1)^2, x^3 + 4x^2 + 7x + 6 + 6k(x^3 - 1)) \\ &= 2^4 \cdot 3^4(6k - 1)^2. \end{aligned}$$

From $q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1) = (q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1), 18 \cdot p^r(q^2 - 1)^2)$ and $(q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1), (q^2 - 1)^2) \mid 2^4 \cdot 3^4(6k - 1)^2$ we have

$$\frac{1}{p^r}(q^3 + 4q^2 + 7q + 6 + 6k(q^3 - 1)) \mid 2^5 \cdot 3^6(6k - 1)^2, \tag{9}$$

where $p^r \| k - 1$. Let $p = 3$, $q = 3^6$, then $2f = 12$, $s(3, f) = 2$. For this set of values the inequality

$$\frac{1}{p^{s(p,f)}} q^3 > 2^5 \cdot 3^6(6 \cdot 2f) \tag{10}$$

holds. From (3.2), for $p \geqslant 3, f \geqslant 6$, and $k \mid 2f$, $p^r \| k - 1$ the following relation

$$\frac{1}{p^r} q^3 > 2^5 \cdot 3^6 \cdot 6k$$

holds. Since in (9) the left side is greater than $(1/p^r)\, 6kq^3$ and the right side less than $2^5 \cdot 3^6 \cdot 6k \cdot 6k$, we have

the left side > the right side.

Thus (9) does not hold.

The remaining cases $q = p^f$ such that there exist k with $k \mid 2f$ and $p \mid k - 1$ are $q = 3^4$, $k = 4$; $q = 3^2$, $k = 4$; $q = 5^3$, $k = 6$; $q = 7^4$, $k = 8$. None of them satisfies condition (2).

By the same method as in the above examples we examine all the $\varphi(1) = 1 + \chi(1) + k\xi_j(1)$, $j = 1, 2, \ldots, 9$, and find that none of them satisfies condition (2). For brevity, we write down only the reduced divisibility relations like (5), (6), (8), (9) and the auxiliary inequalities like (7), (10), and special values $q = p^f$. Those $q = c^g$ with $c < p$ or $g < f$ must be verified individually.

(1) $1+\chi_2(1)+k\xi_1(1)=1+q^6+k(q^4+q^2+1),\ p^r\| k+1,$
$1+q^6+k(q^4+q^2+1) \mid p^r(q^2-1)(q^6-1),$
$\frac{1}{p^r}(1+q^6+k(q^4+q^2+1) \mid 2^4(3k+2)^4,$
$\frac{1}{p^{s(p,f)}} q^6 > 2^4(3\cdot 2f+2)^4,$
$q=3^3.$

(2) $1+\chi_2(1)+k\xi_2(1)=1+q^6+k(q^2-q+1)(q^3-1),\ p^r\| k-1,$
$1+q^6+k(q^2-q+1)(q^3-1) \mid p^r(q^2-1)(q^6-1),$
$\frac{1}{p^r}(1+q^6+k(q^2-q+1)(q^3-1)) \mid 2^8\cdot(3k-1)^2,$
$\frac{1}{p^{s(p,f)}} q^6 > 2^8(3\cdot 2f)^2,$
$q=3^2.$

(3) $1+\chi_2(1)+k\xi_3(1)=1+q^6+k(q^6-1),\ p^r\| k-1,$
$1+q^6+k(q^6-1) \mid p^r(q^2-1)(q^6-1),$
$\frac{1}{p^r}(1+q^6+k(q^6-1)) \mid 2^8,$
$\frac{1}{p^{s(p,f)}} q^6 > 2^8,$
$q \geqslant 5.$

(4) $1+\chi_2(1)+k\xi_4(1)$
$\quad =1+q^6+k(q-1)(q^2-1)(q^3+1),\ p^r\| k+1,$
$1+q^6+k(q-1)(q^2-1)(q^3+1) \mid p^r(q^2-1)(q^6-1),$
$\frac{1}{p^r}(1+q^6+k(q-1)(q^2-1)(q^3+1)) \mid 2^8(3k+1)^2,$
$\frac{1}{p^{s(p,f)}} q^6 > 2^8(3\cdot 2f+1)^2,$
$q=3^2.$

(5) $1+\chi_2(1)+k\xi_5(1)=1+q^6+k(q^2+q+1)(q^3+1),\ p^r\| k+1,$
$1+q^6+k(q^2+q+1)(q^3+1) \mid p^r(q^2-1)(q^6-1),$
$\frac{1}{p^r}(1+q^6+k(q^2+q+1)(q^3+1)) \mid 2^8\cdot(3k+1)^2,$

$$\frac{1}{p^{s(p,f)}} q^6 > 2^8(3 \cdot 2f + 1)^2,$$
$$q = 3^2.$$

(6) $1 + \chi_2(1) + k\xi_6(1)$
$$= 1 + q^6 + k(q^2 + q + 1)(q + 1)(q^3 + 1),\ p^r \| k + 1,$$
$$1 + q^6 + k(q^2 + q + 1)(q + 1)(q^3 + 1) \mid p^r(q^2 - 1)(q^6 - 1),$$
$$\frac{1}{p^r}(1 + q^6 + k(q^2 + q + 1)(q + 1)(q^3 + 1)) \mid 2^8(6k + 1)^2,$$
$$\frac{3}{p^{s(p,f)}} q^6 > 2^8(6 \cdot 2f + 1)^2,$$
$$q = 3^2.$$

(7) $1 + \chi_2(1) + k\xi_7(1)$
$$= 1 + q^6 + k(q - 1)(q^2 - q + 1)(q^3 - 1), p^r \| k + 1,$$
$$1 + q^6 + k(q - 1)(q^2 - q + 1)(q^3 - 1) \mid p^r(q^2 - 1)(q^6 - 1),$$
$$\frac{1}{p^r}(1 + q^6 + k(q - 1)(q^2 - q + 1)(q^3 - 1)) \mid 2^8(6k + 1)^2,$$
$$\frac{2}{p^{s(p,f)}} q^6 > 2^8(6 \cdot 2f + 1)^2,$$
$$q = 3^2.$$

(8) $1 + \chi_2(1) + k\xi_8(1)$
$$= 1 + q^6 + k(q + 1)(q^2 - 1)(q^3 - 1),\ p^r \| k + 1,$$
$$1 + q^6 + k(q + 1)(q^2 - 1)(q^3 - 1) \mid p^r(q^2 - 1)(q^6 - 1),$$
$$\frac{1}{p^r}(1 + q^6 + k(q + 1)(q^2 - 1)(q^3 - 1)) \mid 2^8(3k + 1)^2,$$
$$\frac{2}{p^{s(p,f)}} q^6 > 2^8(3 \cdot 2f + 1)^2,$$
$$q = 3^2.$$

(9) $1 + \chi_2(1) + k\xi_9(1) = 1 + q^6 + k(q^3 \pm 1)\ \ (q \equiv \pm 1, \bmod 3)$
$$1 + q^6 + k(q^3 \pm 1) \mid p^r(q^2 - 1)(q^6 - 1),\ p^r \| k \pm 1,$$
$$\frac{1}{p^r}(1 + q^6 + k(q^3 \pm 1)) \mid 2^8(k \pm 1)^4$$
$$\frac{1}{p^{s(p,f)}} q^6 > 2^8(2f + 1)^4,$$
$$q = 5^2.$$

(10) $1 + t\chi_3(1) + k\xi_1(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q^4 + q^2 + 1), \ t = 1, 2.$$

$$3 + tq(q^4 + q^2 + 1) + 3k(q^4 + q^2 + 1) \mid 3p^r(q^2 - 1)(q^6 - 1),$$
$$p^r \| k + 1,$$

$$\frac{1}{p^r}(3 + tq(q^4 + q^2 + 1) + 3k(q^4 + q^2 + 1)) \mid 3^9((3k + 1)^2 - t^2)^2,$$

$$\frac{1}{p^{s(p,f)}} q^5 > 3^9(3 \cdot 2f + 1)^4,$$

$$q = 3^5.$$

(11) $1 + t\chi_3(1) + k\xi_2(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q^2 - q + 1)(q^3 - 1), t = 1, 2.$$

$$3 + tq(q^4 + q^2 + 1)$$
$$+ 3k(q^2 - q + 1)(q^3 - 1) \mid 3p^r(q^2 - 1)(q^6 - 1), \ p^r \| k - 1,$$

$$\frac{1}{p^r}(3 + tq(q^4 + q^2 + 1)$$
$$+ 3k(q^2 - q + 1)(q^3 - 1)) \mid 3^9(t + 1)^2 (6k + t - 1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^5 > 3^{11}(6 \cdot 2f + 1)^2,$$

$$q = 3^4.$$

(12) $1 + t\chi_3(1) + k\xi_3(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q^6 - 1), \ t = 1, 2.$$

For $t = 2$.

$$3 + 2q(q^4 + q^2 + 1) + 3k(q^6 - 1) \mid 3p^r(q^2 - 1)(q^6 - 1), p^r \| k - 1.$$

$$\frac{1}{p^r}(3 + 2q(q^4 + q^2 + 1) + 3k(q^6 - 1) \mid 3^{11}$$

$$\frac{3}{p^{s(p,f)}} q^6 > 3^{11},$$

$$q = 3^2.$$

For $t = 1$.

$$q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q - 1)$$
$$\times (q^4 + q^2 + 1) \mid 3p^r(q - 1)^2 (q + 1)(q^4 + q^2 + 1),$$

$$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q-1)$$
$$\times (q^4 + q^2 + 1)) \mid 3^8(2k-1),$$
$$\frac{3^2}{p^{s(p,f)}} q^5 > 3^8 \cdot 2 \cdot 2f,$$
$$q = 3^2.$$

(13) $1 + t\chi_3(1) + k\xi_4(1)$
$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q-1)(q^2-1)(q^3+1), t = 1, 2.$$

For $t = 2$.
$$3 + 2q(q^4 + q^2 + 1)$$
$$+ 3k(q-1)(q^2-1)(q^3+1) \mid 3p^r(q^2-1)(q^6-1), p^r \| k+1,$$
$$\frac{1}{p^r}(3 + 2q(q^4 + q^2 + 1)$$
$$+ 3k(q-1)(q^2-1)(q^3+1)) \mid 3^{11}(6k+1)^2,$$
$$\frac{1}{p^{s(p,f)}} q^6 > 3^{11}(36 \cdot 2f + 13),$$
$$q = 3^3.$$

For $t = 1$.
$$q^4 - q^3 + 2q^2 - 2q + 3$$
$$+ 3k(q-1)^2(q^3+1) \mid 3p^r(q-1)^2(q+1)(q^4+q^2+1),$$
$$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q-1)^2(q^3+1)) \mid 3^8(6k+1)^2,$$
$$\frac{1}{p^{s(p,f)}} q^5 > 3^8(36 \cdot 2f + 13),$$
$$q = 3^3.$$

(14) $1 + t\chi_3(1) + k\xi_5(1)$
$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q^3+1)(q^2+q+1), t = 1, 2.$$

For $t = 2$.
$$3 + 2q(q^4 + q^2 + 1)$$
$$+ 3k(q^2+q+1)(q^3+1) \mid 3p^r(q^2-1)(q^6-1), p^r \| k+1,$$
$$\frac{1}{p^r}(3 + 2q(q^4 + q^2 + 1) + 3k(q^2+q+1)(q^3+1)) \mid 3^{11}(2k+1)^2,$$

$$\frac{3}{p^{s(p,f)}} q^5 > 3^{11}(4 \cdot 2f + 5),$$

$q = 3^3$.

For $t = 1$.

$$q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^4 + q^2 + 1) \mid 3p^r(q-1)^2 (q+1) \times (q^4 + q^2 + 1),$$

$$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^4 + q^2 + 1)) \mid 3^8(3k+1)^2 \times (k+1),$$

$$\frac{3}{p^{s(p,f)}} q^4 > 3^8(9 \cdot 2f + 7)(2f + 1),$$

$q = 3^4$.

(15) $1 + t\chi_3(1) + k\xi_6(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q+1)(q^2 + q + 1)(q^3 + 1),$$

$t = 1, 2$.

For $t = 2$.

$$3 + 2q(q^4 + q^2 + 1) + 3k(q+1)(q^2 + q + 1)(q^3 + 1) \mid 3p^r(q^2 - 1)(q^6 - 1),$$

$$p^r \| k + 1,$$

$$\frac{1}{p^r}(3 + 2q(q^4 + q^2 + 1) + 3k(q+1)(q^2 + q + 1)(q^3 + 1)) \mid 3^{11} \times (4k+1)^2,$$

$$\frac{3}{p^{s(p,f)}} q^6 > 3^{11}(16 \cdot 2f + 9),$$

$q = 3^3$.

For $t = 1$.

$$q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 + q + 1)(q^3 + 1) \mid 3p^r(q-1)^2 (q+1)(q^4 + q^2 + 1),$$

$$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 + q + 1)(q^3 + 1)) \mid 3^8 \times (6k+1)^2,$$

$$\frac{1}{p^{s(p,f)}} \cdot 3q^5 > 3^8(36 \cdot 2f + 13),$$

$q = 3^3$.

(16) $1 + t\chi_3(1) + k\xi_7(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q-1)(q^2 - q + 1)(q^3 - 1),$$
$$t = 1, 2.$$

$$3 + tq(q^4 + q^2 + 1) + 3k(q-1)(q^2 - q + 1)(q^3 - 1) \mid 3p^r(q^2 - 1)(q^6 - 1),$$
$$p^r \| k + 1,$$

$$\frac{1}{p^r}(3 + tq(q^4 + q^2 + 1) + 3k(q-1)(q^2 - q + 1)(q^3 - 1)) \mid 3^9(1+t)^2\,(12k - t + 1)^2,$$

$$\frac{1}{p^{s(p,f)}} \cdot 3q^6 > 3^{11}(12 \cdot 2f)^2,$$

$$q = 3^4.$$

(17) $1 + t\chi_3(1) + k\xi_8(1)$

$$= 1 + \frac{t}{3} q(q^4 + q^2 + 1) + k(q+1)(q^2 - 1)(q^3 - 1),\ t = 1, 2.$$

For $t = 2$.

$$3 + 2q(q^4 + q^2 + 1) + 3k(q+1)(q^2 - 1)(q^3 - 1) \mid 3p^r(q^2 - 1)(q^6 - 1),\ p^r \| k + 1,$$

$$\frac{1}{p^r}(3 + 2q(q^4 + q^2 + 1) + 3k(q+1)(q^2 - 1)(q^3 - 1)) \mid 3^{11}(6k + 1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^6 > 3^{11}(36 \cdot 2f + 13),$$

$$q = 3^3.$$

For $t = 1$.

$$q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 - 1)(q^3 - 1) \mid 3p^r(q-1)^2\,(q+1)(q^4 + q^2 + 1),$$
$$p^r \| k + 1,$$

$$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 - 1)(q^3 - 1)) \mid 3^8(6k + 1)^2,$$

$$\frac{3}{p^{s(p,f)}} q^5 > 3^8(36 \cdot 2f + 13),$$

$$q = 3^3.$$

(18) $1 + t\chi_3(1) + k\xi_9(1) = 1 + \frac{t}{3} q(q^2 + q + 1) + k(q^3 \pm 1)$

$(q \equiv \pm 1, \bmod 3)$

For $t = 2$ or $t = 1$ with $q \equiv -1 \pmod 3$.

$3 + tq(q^4 + q^2 + 1) + 3k(q^3 \pm 1) \mid 3p^r(q^2 - 1)(q^6 - 1), p^r \| k \pm 1,$

$\frac{1}{p^r}(3 + tq(q^4 + q^2 + 1)$

$+ 3k(q^3 \pm 1)) \mid 3^9(2k \pm 1)^2 (1 \mp t)^2 (2k + t \pm 1)^2,$

$\frac{1}{p^{s(p,f)}} q^5 > 3^9(2 \cdot 2f + 1)^2 (2 \cdot 2f + 3)^2 \cdot 5$

$q = 5^3.$

For $t = 1$ with $q \equiv 1 \pmod 3$

$q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 - q + 1) \mid 3p^r(q - 1)(q^6 - 1),$

$p^r \| k + 1,$

$\frac{1}{p^r}(q^4 - q^3 + 2q^2 - 2q + 3 + 3k(q^2 - q + 1)) \mid 3^8(k + 1)^3$

$\times (2k + 1)^2,$

$\frac{1}{2p^{s(p,f)}} q^4 > 3^8(2f + 1)^3 (2 \cdot 2f + 1)^2,$

$q = 5^4.$

(19) $1 + \chi_5(1) + k\xi_1(1)$

$= 1 + \frac{q}{2}(q + 1)^2 (q^2 - q + 1) + k(q^4 + q^2 + 1),$

$q^3 - q + 2 + 2k(q^2 - q + 1) \mid 2p^r(q^2 - 1)^2 (q^2 - q + 1), p^r \| k + 1,$

$\frac{1}{p^r}(q^3 - q + 2 + 2k(q^2 - q + 1)) \mid 2^5(k + 1)^2)3k + 1)^2,$

$\frac{1}{p^{s(p,f)}} q^3 > 2^5(2f + 1)^2 (3 \cdot 2f + 1)^2,$

$q = 3^6.$

(20) $1 + \chi_5(1) + k\xi_2(1)$

$= 1 + \frac{q}{2}(q + 1)^2 (q^2 - q + 1) + k(q^2 - q + 1)(q^3 - 1),$

$q^3 - q + 2 + 2k(q - 1)(q^2 - q + 1) \mid 2p^r(q^2 - 1)^2 (q^2 - q + 1),$

$p^r \| k + 1,$

$$\frac{1}{p^r}(q^3-q+2+2k(q-1)(q^2-q+1)) \mid 2^5(6k-1)^2,$$

$$\frac{1}{p^{s(p,f)}}\cdot 2q^3 > 2^5\cdot(36\cdot 2f),$$

$$q=3^3.$$

(21) $$1+\chi_5(1)+k\xi_3(1)=1+\frac{q}{2}(q+1)^2(q^2-q+1)+k(q^6-1),$$

$$q^3-q+2+2k(q^2-1)(q^2-q+1) \mid 2p^r(q^2-1)^2(q^2-q+1),$$

$$p^r \| k-1,$$

$$\frac{1}{p^r}(q^3-q+2+2k(q^2-1)(q^2-q+1)) \mid 2^5,$$

$$q^4>2^5,$$

$$q\geqslant 5.$$

(22) $$1+\chi_5(1)+k\xi_4(1)$$

$$=1+\frac{q}{2}(q+1)^2(q^2-q+1)+k(q-1)(q^2-1)(q^3+1),$$

$$2+q(q+1)^2(q^2-q+1)$$
$$+2k(q-1)(q^2-1)(q^3+1) \mid 2p^r(q^2-1)^2(q^2+q+1),$$
$$p^r \| k+1,$$

$$\frac{1}{p^r}(2+q(q+1)^2(q^2-q+1)$$
$$+2k(q-1)(q^2-1)(q^3+1)) \mid 2^9\cdot 3^4\cdot k^2,$$

$$\frac{1}{p^{s(p,f)}}q^6>2^9\cdot 3^4\cdot 2f,$$

$$q=3^2.$$

(23) $$1+\chi_5(1)+k\xi_5(1)$$

$$=1+\frac{q}{2}(q+1)^2(q^2-q+1)+k(q^3+1)(q^2+q+1),$$

$$q^3-q+2+2k(q^3+1) \mid 2p^r(q^2-1)^2(q^2-q+1), p^r \| k+1,$$

$$\frac{1}{p^r}(q^3-q+2+2k(q^3+1)) \mid 2^5(2k+1)^2,$$

$$\frac{1}{p^{s(p,f)}}q^3>2^4\cdot(4\cdot 2f+5),$$

$$q=3^3.$$

(24) $1 + \chi_5(1) + k\xi_6(1)$

$$= 1 + \frac{q}{2}(q+1)^2(q^2-q+1) + k(q+1)(q^2+q+1) \times (q^3+1),$$

$$q^3 - q + 2 + 2k(q+1)^2(q^2-q+1) \mid 2p^r(q^2-1)^2(q^2-q+1), \quad p^r \| k+1,$$

$$\frac{1}{p^r}(q^3 - q + 2 + 2k(q+1)^2(q^2-q+1)) \mid 2^5(4k+1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^4 > 2^4(16 \cdot 2f + 9),$$

$$q = 3^2.$$

(25) $1 + \chi_5(1) + k\xi_7(1)$

$$= 1 + \frac{q}{2}(q+1)^2(q^2-q+1) + k(q-1)(q^2-q+1) \times (q^3-1),$$

$$q^3 - q + 2 + 2k(q-1)^2(q^2-q+1) \mid 2p^r(q^2-1)^2(q^2-q+1), \quad p^r \| k+1,$$

$$\frac{1}{p^r}(q^3 - q + 2 + 2k(q-1)^2(q^2-q+1)) \mid 2^5(12k+1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^4 > 2^5(144 \cdot 2f + 25),$$

$$q = 3^3.$$

(26) $1 + \chi_5(1) + k\xi_8(1)$

$$= 1 + \frac{q}{2}(q+1)^2(q^2-q+1) + k(q+1)(q^2-1)(q^3-1),$$

$$q^3 - q + 2 + 2k(q^2-1)^2 \mid 2p^r(q^2-1)^2(q^2-q+1), p^r \| k+1,$$

$$\frac{1}{p^r}(q^3 - q + 2 + 2k(q^2-1)^2) \mid 2^5(6k+1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^4 > 2^4(36 \cdot 2f + 13),$$

$$q = 3^3.$$

(27) $1 + \chi_5(1) + k\xi_9(1) = 1 + \frac{q}{2}(q+1)^2(q^2-q+1) + k(q^3 \pm 1)$

$$(q \equiv \pm 1, \bmod 3),$$

For $q \equiv -1 \pmod 3$.

$q^3 - q + 2 + 2k(q-1) \mid 2p^r(q^2-1)^2(q^2-q+1), p^r \| k-1,$

$\frac{1}{p^r}(q^3 - q + 2 + 2k(q-1)) \mid 2^5(2k-1)^4,$

$\frac{1}{p^{s(p,f)}} q^3 > 2^5(2 \cdot 2f)^4,$

$q = 5^4.$

For $q \equiv 1 \pmod 3$.

$2 + q(q+1)^2(q^2-q+1) + 2k(q^3+1) \mid 2p^r(q^2-1)(q^6-1),$

$p^r \| k+1,$

$\frac{1}{p^r}(2 + q(q+1)^2(q^2-q+1) + 2k(q^3+1)) \mid 2^{11}(3+2k)^2 k^2,$

$\frac{1}{p^{s(p,f)}} q^5 > 2^{11}(3 + 2f \cdot 2)^2(2f)^2,$

$q = 5^3.$

(28) $1 + \chi_6(1) + k\xi_1(1)$

$$= 1 + \frac{q}{6}(q+1)^2(q^2+q+1) + k(q^4+q^2+1),$$

$q^3 + 4q^2 + 7q + 6 + 6k(q^2+q+1) \mid 18p^r(q^2-1)^2, p^r \| k+1,$

$\frac{1}{p^r}(q^3 + 4q^2 + 7q + 6 + 6k(q^2+q+1)) \mid 2^5 \cdot 3^6(k+1)^2$
$\times (3k+1)^2,$

$\frac{1}{p^{s(p,f)}} q^3 > 2^5 \cdot 3^6(2f+1)^2(3 \cdot 2f+1)^2,$

$q = 3^8.$

(29) $1 + \chi_6(1) + k\xi_2(1)$

$$= 1 + \frac{q}{6}(q+1)^2(q^2+q+1) + k(q^2-q+1)(q^3-1),$$

$q^3 + 4q^2 + 7q + 6 + 6k(q^3-1) \mid 18p^r(q^2-1)^2, p^r \| k+1,$

$\frac{1}{p^r}(q^3 + 4q^2 + 7q + 6 + 6k(q^3-1)) \mid 2^5 \cdot 3^6(6k-1)^2,$

$\frac{1}{p^{s(p,f)}} q^3 > 2^5 \cdot 3^6 \cdot 6 \cdot 2f,$

$q = 3^6.$

(30) $1+\chi_6(1)+k\xi_3(1)=1+\dfrac{q}{6}(q+1)^2(q^2+q+1)+k(q^6-1)$,

$q^3+4q^2+7q+6+6k(q^2-1)(q^2+q+1) \mid 18p^r(q^2-1)^2$,

$p^r \| k-1$,

$\dfrac{1}{p^r}(q^3+4q^2+7q+6+6k(q^2-1)(q^2+q+1)) \mid 2^5\cdot 3^6$,

$6q^4>2^5\cdot 3^6$,

$q\geqslant 9$.

(31) $1+\chi_6(1)+k\xi_4(1)$

$$=1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q-1)(q^2-1)(q^3+1),$$

$q^3+4q^2+7q+6+6k(q^2-1)^2 \mid 6p^r(q^2-1)^2(q^2+q+1)$,

$p^r \| k+1$,

$\dfrac{1}{p^r}(q^3+4q^2+7q+6+6k(q^2-1)^2) \mid 2^5\cdot 3^7(6k+1)^2$,

$\dfrac{1}{p^{s(p,f)}}q^4>2^5\cdot 3^7(6\cdot 2f+3)$,

$q=3^5$.

(32) $1+\chi_6(1)+k\xi_5(1)$

$$=1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q^2+q+1)(q^3+1),$$

$q^3+4q^2+7q+6+6k(q^2+q+1)(q+1) \mid 18p^r(q^2-1)^2$,

$p^r \| k+1$,

$\dfrac{1}{p^r}(q^3+4q^2+7q+6+6k(q^2+q+1)(q+1)) \mid 2^5\cdot 3^6(2k+1)^2$

$\dfrac{1}{p^{s(p,f)}}q^3>2^4\cdot 3^5(4\cdot 2f+5)$,

$q=3^5$.

(33) $1+\chi_6(1)+k\xi_6(1)$

$$=1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q+1)(q^2+q+1)$$

$$\times(q^3+1),$$

$q^3+4q^2+7q+6+6k(q+1)^2(q^2+q+1) \mid 18p^r(q^2-1)^2$,

$p^r \| k+1$,

$$\frac{1}{p^r}(q^3+4q^2+7q+6+6k(q+1)^2(q^2+q+1)) \mid 2^5 \cdot 3^6(4k+1)^2,$$

$$\frac{1}{p^{s(p,f)}}q^4 > 2^4 \cdot 3^5(16 \cdot 2f+9),$$

$$q = 3^4.$$

(34) $1+\chi_6(1)+k\xi_7(1)$

$$= 1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q-1)(q^3-1) \times (q^2-q+1),$$

$$q^3+4q^2+7q+6+6k(q-1)^2(q^2+q+1) \mid 18p^r(q^2-1)^2, \quad p^r \| k+1,$$

$$\frac{1}{p^r}(q^3+4q^2+7q+6+6k(q-1)^2(q^2+q+1)) \mid 3^6 \cdot 2^5(12k+1)^2,$$

$$\frac{5}{p^{s(p,f)}}q^4 > 3^6 \cdot 2^5(144 \cdot 2f+25),$$

$$q = 3^5.$$

(35) $1+\chi_6(1)+k\xi_8(1)$

$$= 1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q+1)(q^2-1)(q^3-1),$$

$$6+q(q+1)^2(q^2+q+1) + 6k(q^2-1)^2(q^2+q+1) \mid 18p^r(q^2-1)^2(q^2-q+1), \quad p^r \| k+1,$$

$$\frac{1}{p^r}(6+q(q+1)^2(q^2+q+1)+6k(q^2-1)^2 \times (q^2+q+1)) \mid 2^9 \cdot 3^{12}k^2,$$

$$\frac{1}{p^{s(p,f)}}q^6 > 2^8 \cdot 3^{11} \cdot 2f,$$

$$q = 3^4.$$

(36) $1+\chi_6(1)+k\xi_9(1) = 1+\frac{q}{6}(q+1)^2(q^2+q+1)+k(q^3 \pm 1)$

$(q \equiv \pm 1, \bmod 3),$

For $q \equiv -1 \pmod 3$.

$$6 + q(q+1)^2(q^2+q+1) + 6k(q^3-1) \mid 6p^r(q^2-1)^2 \times (q^2-q+1), p^r \| k-1,$$

$$\frac{1}{p^r}(6 + q(q+1)^2(q^2+q+1) + 6k(q^3-1)) \mid 2^9 \cdot 3^9 k^2(2k-1)^2,$$

$$\frac{1}{p^{s(p,f)}} q^5 > 2^9 \cdot 3^9 (2f)^2 (2 \cdot 2f)^2,$$

$$q = 5^4.$$

For $q \equiv 1 \pmod 3$.

$$q^3 + 4q^2 + 7q + 6 + 6k(q+1) \mid 6p^r(q^2-1)^2(q^2+q+1), \quad p^r \| k+1,$$

$$\frac{1}{p^r}(q^3 + 4q^2 + 7q + 6 + 6k(q+1)) \mid 2^5 \cdot 3^5 (3+2k)^2(1+2k)^2,$$

$$\frac{1}{p^{s(p,f)}} q^3 > 2^5 \cdot 3^5 (3 + 2 \cdot 2f)^2 (1 + 2 \cdot 2f)^2,$$

$$q = 5^5.$$

From the argument in the first sections, we reach:

Conclusion. If $\varphi^* = 1_{K^*}^{G^*}$ is a faithful primitive permutation representation of rank 3 of G^*, then the possibilities of $\varphi(1) = 1 + \chi(1) + k\xi(1)$ are

(i) $\varphi(1) = q^6$ or $2q^6$,

(ii) $\varphi(1) = q^3(q^3+1)$, $\chi(1) = q^6$, $k = 1$, $\xi(1) = q^3 - 1$ and $q \equiv -1 \pmod 3$,

(iii) $\varphi(1) = 156$, $\chi(1) = \chi_3(1) = \chi_4(1) = 91$, $k = 1$, $\xi(1) = 64$ and $q = 3$,

(iv) $\varphi(1) = 273$, $\chi(1) = \chi_6(1) = 168$, $k = 1$, $\xi(1) = 104$ and $q = 3$,

(v) $\varphi(1) = 378$, $\chi(1) = \chi_5(1) = 104$, $k = 1$, $\xi(1) = 273$ and $q = 3$,

(vi) $\varphi(1) = 351$, $\chi(1) = \chi_6(1) = 168$, $k = 2$, $\xi(1) = 91$ and $q = 3$,

(vii) $\varphi(1) = 351$, $\chi(1) = \chi_6(1) = 168$, $k = 1$, $\xi(1) = 182$ and $q = 3$,

(viii) $\varphi(1) = 7^2 \cdot 9 \cdot 43$, $\chi(1) = 2^5 \cdot 8 \cdot 19$, $k = 1$, $\xi(1) = 2 \cdot 3^2 \cdot 19 \cdot 43$ and $q = 7$.

4. The Existence of the Subgroup K

We set up several theorems about this.

THEOREM 4.1. *None of subgroups K of $G_2(q)$ satisfies*

$$(q^2 - q + 1)(q^2 + q + 1) \mid |K| \qquad \text{and} \qquad p \nmid |K|.$$

Proof. The proof is almost the same as an argument of Seitz for another proposition [14, p. 35]. In order to describe the proof in detail we give Lemmas 4.2–4.5.

Suppose the result is false and K is a counterexample.

LEMMA 4.2. *There is a prime divisor γ of one of $q^2 \pm q + 1$ such that $\gamma \geqslant 13$ for $q \geqslant 5$ and if* none of $\gamma > 13$, *then 13^2 divides one of $q^2 \pm q + 1$.*

Proof. Directly checking for any q we have $q^2 \pm q + 1 \neq 0$ (mod 5, 9, 11). Then 5, 9, 11 cannot divide $q^2 \pm q + 1$. Hence both $q^2 \pm q + 1$ are divisible by at most the first power of 3 and at least one of $q^2 \pm q + 1$ has no divisor 5, 7, and 11, since $(q^2 + q + 1, q^2 - q + 1) = 1$. This number must have a prime divisor $\gamma \geqslant 13$.

Suppose it has neither prime divisor $\gamma > 13$ nor divisor 13^2, then it equals 13 or 39. For $q \geqslant 5$, this is not in the case.

LEMMA 4.3. *K contains a normal cyclic subgroup R of order γ, where $\gamma \geqslant 13$ is a prime and $\gamma \mid (q^2 + q + 1)(q^2 - q + 1)$.*

Proof. As $(q^2 + q + 1)(q^2 - q + 1) \mid |K|$. $|K|$ has γ in Lemma 4.2 as a divisor. Since $(q^2 - q + 1)(q^2 + q + 1)$ and $(q^2 - 1)$ have no common prime divisor other than 3 and $|G| = q^6(q^2 - 1)^2 (q^2 + q + 1)(q^2 - q + 1)$, $|K|$ and $|G|$ contain the same power of γ. Thus, a γ-Sylow subgroup of K is one of G. From [2, Section 1] and [2, (2.7)], it is contained in a maximal torus of G of order $q^2 + q + 1$ or $q^2 - q + 1$ and the two tori are cyclic, thus so is a γ-Sylow subgroup.

We claim that K contains a normal subgroup of order γ. $G_2(q)$ has a faithful representation V of dimension 7, the characteristic of the basic field is p and $p \nmid |K|$. It is an ordinary representation of K.

For $q = 3$, $q^2 + q + 1 = 13$ and $q^2 - q + 1 = 7$, so $\gamma = 13$; V is of dimension 7, $13 > 7 + 2$. From [6], $G_2(3)$ has no element whose order is divisible by both a prime divisor of $q^2 + q + 1$ and a prime divisor of $q^2 - q + 1$, then $Z(K) = 1$. By [7, Theorem 1] the 13-Sylow subgroup of K is normal in K or K is isomorphic to $PsL(2, 13)$. The latter case is out as $3 \mid |PSL(2, 13)|$ and $|K| \mid (3^2 - 1)^2 \cdot 7 \cdot 13$. The 13-Sylow subgroup of K meets the requirement of our lemma.

For $q \geqslant 5$ and $\gamma > 13$, or course $\gamma > 15$ and $\frac{1}{2}(\gamma - 1) > 7$. By [8, Theorem 1] the γ-Sylow subgroup of K is normal in K. And it is also cyclic, then its cyclic subgroup of order γ is still normal in K.

For $q \geqslant 5$ and $\gamma = 13$, of course $13 - 1 > 7$. By the main theorem of [9] the 13-Sylow subgroup of K contains a subgroup of index $\leqslant 13$ which is normal in K. Since $13^2 \mid |K|$, the subgroup is a nontrivial 13-group which is also cyclic. Then its cyclic subgroup of order 13 is still normal in K.

LEMMA 4.4. *Let R be a subgroup of $GL(l, q)$ of order γ, where γ is a prime and $(\gamma, q) = 1$. If $\gamma \nmid q^l - 1$, then R is reducible.*

Proof. We know $|GL(n, q)| = q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$. Since $\gamma \mid |GL(l, q)|$ and $\gamma \nmid q^l - 1$, $|GL(l, q)|$ and $|GL(l - 1, q)|$ contains the same power of γ as their divisors. The γ-Sylow subgroup of $GL(l - 1, q)$ is one of $GL(l, q)$. R is a γ-subgroup of $GL(l, q)$, thus it is conjugate to a subgroup of $GL(l - 1, q)$, namely it is reducible.

LEMMA 4.5. *Restricting the 7-dimension faithful representation V of G to R, where R is the normal cyclic subgroup of G or order γ as in the Lemma 4.3, and denoting the restriction by V_R, then the dimensions of nontrivial irreducible constituents of V_R are equal to 3 or 6.*

Proof. Let l be the dimension of an nontrivial irreducible constituent of V_R. This representation of R is a subgroup of $GL(l, q)$. By Lemma 4.4, $\gamma \mid q^l - 1$. And from $\gamma \mid (q^2 + q + 1)(q^2 - q + 1)$, we have $\gamma \mid q^6 - 1$. Clearly $\gamma \nmid q^2 - 1$. Thus the order of $q \pmod{\gamma}$ is 3 or 6. Hence $3 \mid l$. And since $l \leqslant 7$, l is equal to 3 or 6.

Proof of Theorem 4.1. Let V_1 be a nontrivial irreducible constituent of V_R. If $\dim V_1 = 3$, then R is isomorphic to a subgroup of $GL(3, q)$. Thus $\gamma \| GL(3, q)|$. Since $q^2 - q + 1$ and $|GL(3, q)| = q^3(q^3 - 1)(q^2 - 1)(q - 1)$ have no common prime divisor other than 3 and $\gamma \mid (q^2 - q + 1)(q^2 + q + 1)$, $\gamma \mid q^2 + q + 1$. By using [14, (1.7)], there exists a prime divisor s of $q^2 - q + 1$ such that $s \nmid q^l - 1$ with $l \leqslant 5$. Let g_0 be an element of order s of K. $\langle R, g_0 \rangle$ is order $s\gamma$ since g_0 normalizes R. Since $s \nmid |GL(l, q)|$ for $l \leqslant 5$ and $s \mid q^6 - 1$, the nontrivial irreducible constituent of $V_{\langle g_0 \rangle}$ is just of dimension 6 and $V_{\langle g_0 \rangle}$ has no invariant subspace of $\langle g_0 \rangle$ of dimension other than 6 or 1.

Let $V_1^{g_0} = \{\alpha^{g_0} \mid \alpha \in V_1\}$. It is still an irreducible constituent of R since g_0 normalizes R. If $V_1^{g_0} \cap V_1 \neq \{0\}$, then $V_1^{g_0} = V_1$ and $\langle g_0 \rangle$ has an invariant subspace of dimension 3. This is impossible, so $V_1^{g_0} \cap V_1 = (0)$ and $V_1^{g_0} \oplus V_1 = V_0$. V_0 is of dimension 6, so $V_0 \cap V_1^{g_0^i} \neq (0)$ for all i. Then $V_1^{g_0^i} \subseteq V_0$ and V_0 is irreducible under $\langle R, g_0 \rangle$. From $R \lhd \langle R, g_0 \rangle$ and [3, (53, 17)], we have $2 \mid |\langle R, g_0 \rangle : R|$, namely $2 \mid s$. But s is odd, this is impossible.

Thus dim $V_1 = 6$. From $R \lhd K$ and Clifford theorem, it is easy to see that V_1 is invariant and irreducible under K. Let ζ be a proper value of the representation matrix of an element A of R, where $A \neq 1$. We have $\zeta^\gamma = 1$. Since V_1 is irreducible under R, $\zeta \notin GF(q)$. Note that $\gamma | q^6 - 1$ and $\gamma \nmid q^2 - 1$, thus $GF(q)(\zeta) = GF(q^3)$ or $GF(q)(\zeta) = CF(q^6)$. Hence all the conditions in [12, Lemma 1.1] hold, we obtain

$$|K|/|C_K(R)| \,\big|\, 6.$$

Namely, $|K| \,\big|\, 6\,|C_K(R)|$ and we have

$$(q^2 - q + 1)(q^2 + q + 1) \,\big|\, 3\,|C_K(R)|.$$

Since $q^2 \pm q + 1 > 3$, $|C_K(R)|$ is divisible by both a divisor of $q^2 - q + 1$ and a divisor of $q^2 + q + 1$. From [2, 5], $G_2(q)$ has no element which satisfies the property. This contradiction completes the proof of Theorem 1.

COROLLARY 4.6. *There exists no subgroup K of G_2 such that the degree of $1_K^{G_2}$ is $2q^6$ or q^6.*

THEOREM 4.7. *There exists no subgroup K of G_2 such that $1_K^{G_2}$ is of rank* 3 *and its irreducible constituents are of dimension* 1, q^6, $q^3 - 1$, *respectively.*

Proof. Suppose such a subgroup K exists. We have

$$|1_K^{G_2}| = 1 + q^6 + q^3 - 1 = q^3(q^3 + 1),$$

then

$$|K| = |G_2|/|1_K^{G_2}| = q^3(q^3 - 1)(q^2 - 1).$$

Let $1, K_1, K_2$ be the lengths of the orbits of K, then $K_i \,\big|\, |K|$, $(i = 1, 2)$ and

$$|1_K^{G_2}| = 1 + K_1 + K_2 = q^3(q^3 + 1).$$

From [11, Theorem 1] of Frame,

$$\frac{q^3(q^3 + 1)\,K_1 K_2}{q^6(q^3 - 1)} = (\text{integer})^2.$$

Thus $q^3 | K_1 K_2$.

We note that $1 + K_1 + K_2 = q^3(q^3 + 1)$, then only one of K_1 and K_2 is divisible by q. Hence we can assume $q^3 | K_1$ and $q^3 | K_2 + 1$. From $K_2 \,\big|\, |K|$, we have $K_2 | (q^3 - 1)(q^2 - 1)$, thus $K_2 < q^5 - 1$. We can set $K_2 + 1 = aq^4 + bq^3$, where $0 \leqslant a$, $b \leqslant q - 1$ and a, b are not all zero. Then $K_2 =$

$aq^4 + bq^3 - 1$. Assume $(q^3 - 1)(q^2 - 1) = tK_2$, where t is a positive integer. As $K_2 = -1 \pmod q$, $t \equiv -1 \pmod q$ holds. Since $t < q^5 - 1$, similarly we set $t = cq^4 + dq^3 + eq^2 + fq - 1$, where $0 \leqslant c, d, e, f \leqslant q - 1$, and c, d, e, f are not all zero.

We consider two cases.

Case 1. $a \neq 0$. If c, d, e, are not all zero, then $t \geqslant q^2 - 1$. Thus

$$K_2 t \geqslant (q^4 - 1)(q^2 - 1) > (q^3 - 1)(q^2 - 1),$$

this is a contradiction. We have $c = d = e = 0$.

If $f \geqslant 2$,

$$K_2 t > (q^4 - 1)(2q - 1) = 2q^5 - q^4 - 2q + 1 > q^5 + 1 > (q^3 - 1)(q^2 - 1).$$

This is impossible, then $f = 1$, $t = q - 1$, and

$$K_2 = (q^3 - 1)(q^2 - 1)/(q - 1) = (q^3 - 1)(q + 1).$$

Thus $K_1 = q^3(q^3 + 1) - 1 - K_2 = q^6 - q^4 + q$ and $q^3 \nmid K_1$. This is still impossible and $a \neq 0$ is not in the case.

Case 2. $a = 0$, $b \neq 0$. If one of the following cases holds:

(i) c, d are not all zero,

(ii) $c = d = 0$, $e > 1$,

(iii) $c = d = 0$, $e = 1$, $f > 0$,

(iv) $c = d = f = 0$, $e = 1$, $b > 1$,

then

$$K_2 t = (bq^3 - 1)(cq^4 + dq^3 + eq^2 + fq - 1) > (q^3 - 1)(q^2 - 1).$$

These cannot hold. Then $c = d = 0$, $e = b = 1$, and $f = 0$ or $c = d = e = 0$, $f > 0$.

For the former case, $K_2 = q^3 - 1$ and

$$K_1 = q^3(q^3 + 1) - 1 - K_2 = q^6,$$

then $K_1 \nmid |K|$. This is impossible. For the latter case,

$$\begin{aligned} tK_2 &= (fq - 1)(bq^3 - 1) = bfq^4 - bq^3 - fq + 1 \\ &= (q^3 - 1)(q^2 - 1) = q^5 - q^3 - q^2 + 1. \end{aligned}$$

We have $q^2 | fq$, then $q | f$. However, $0 < f \leqslant q - 1$, this is impossible. All the contradictions prove the theorem.

THEOREM 4.8. *There exists no subgroup K of $G_2(q)$ such that $1_K^{G_2}$ is of rank 3 and one of the following holds:*

(1) $\varphi(1)=156, \chi(1)=91$, $\xi(1)=64$ *and* $q=3$,

(2) $\varphi(1)=378, \chi(1)=104$, $\xi(1)=273$ *and* $q=3$,

(3) $\varphi(1)=273, \chi(1)=104$, $\xi(1)=168$ *and* $q=3$,

(4) $\varphi(1)=7^2\cdot 9\cdot 43$, $\chi(1)=7\cdot 2^5\cdot 19$, $\xi(1)=2\cdot 3^2\cdot 19\cdot 43$ *and* $q=7$,

where $1_K^{G_2}=\varphi=1+\chi+\xi$.

Proof. Suppose the results are false and K designates a counterexample. Let the lengths of the transitive domains of K be 1, K_1, K_2, then $K_i \,\|\, |K|$, $(i=1,2)$ and $\varphi(1)=1+K_1+K_2$. We know

$$|G_2(3)|=3^6(3^6-1)(3^2-1)=2^6\cdot 3^6\cdot 7\cdot 13$$

and

$$|G_2(7)|=7^6(7^6-1)(7^2-1)=2^8\cdot 3^3\cdot 7^6\cdot 19\cdot 43.$$

(1) We have $|K|=|G_2(3)|/\varphi(1)=2^4\cdot 3^5\cdot 7$. From [11, Theorem 1] of Frame

$$\frac{\varphi(1)\,K_1K_2}{\chi(1)\,\xi(1)}=\frac{13\cdot 3\cdot 4\cdot K_1\cdot K_2}{13\cdot 7\cdot 2^6}=\frac{3\cdot K_1\cdot K_2}{7\cdot 2^4}=(\text{integer})^2.$$

Thus $3\cdot 7\cdot 2^4|K_1K_2$. We can assume $7|K_1$ and $(7,K_2)=1$ since $K_1+K_2=156-1=31\cdot 5$ is prime to 7. Similarly 3, 2^4 divide only one of K_1 and K_2. Hence the possibilities of K_2 and $K_1=155-K_2$ are in

K_2	1	3	3^2	3^3	3^4	2^4	$2^4\cdot 3$
$K_1=155-K_2$	154	152	146	128	74	139	107.

Since K_1 is not a divisor of $|K|$ in all cases, these are all impossible.

(2) In this case $|K|=2^6\cdot 3^3\cdot 7\cdot 13/2\cdot 3^3\cdot 7=2^5\cdot 3^3\cdot 13$. From the Frame's theorem

$$\frac{\varphi(1)\,K_1K_2}{\chi(1)\,\xi(1)}=\frac{2\cdot 3^3\cdot 7\cdot K_1K_2}{8\cdot 13\cdot 3\cdot 13\cdot 7}=\frac{3^2K_1K_2}{4\cdot 13^2}=(\text{integer})^2$$

and $4\cdot 13^2|K_1K_2$. Since $K_1+K_2=378-1=13\cdot 29$, $13|K_i(i=1,2)$ and we can assume $4|K_1$, $(2,K_2)=1$. The possibilities of K_1 and K_2 are in

K_2	13	$3\cdot 13$	$3^2\cdot 13$	$3^3\cdot 13$
$K_1=378-K_2$	$28\cdot 13$	$26\cdot 13$	$20\cdot 13$	$2\cdot 13.$

Since $28 \cdot 13$, $26 \cdot 13$ and $20 \cdot 13$ are not divisors of $|K|$ and $2 \cdot 13$ is not divisible by 4, these are all impossible.

(3) We have $|K| = 3^6 \cdot 2^6 \cdot 7 \cdot 13/273 = 3^5 \cdot 2^6$. From Fram's theorem

$$\frac{273 \cdot K_1 K_2}{8 \cdot 13 \cdot 2^3 \cdot 3 \cdot 7} = \frac{K_1 K_2}{2^6} = (\text{integer})^2$$

and $2^6 | K_1 K_2$. Since $K_1 + K_2 = 273 - 1 = 272 = 2^3 \cdot 34$, $2^3 | K_i$ $(i = 1, 2)$. If $3 \nmid K_i$ $(i = 1, 2)$, then $K_i | 2^6$ since $K_i | 3^5 \cdot 2^6$. Thus $K_i \leqslant 2^6 = 64$ $(i = 1, 2)$ and $K_1 + K_2 < 272$. This is impossible. Since $3 \nmid 272$, $K_1 + K_2 = 272$, we can assume $3 | K_1$ and $(3, K_2) = 1$. Hence the possibilities of K_1 and K_2 are

K_2	2^3	2^4	2^5	2^6
$K_1 = 272 - K_2$	264	256	240	208.

Since $264 \nmid 2^6 \cdot 3^5$, $240 \nmid 2^6 \cdot 3^5$, and $3 \nmid 256$, $3 \nmid 208$, these are impossible.

(4) In this case $|K| = |G_2(7)|/\varphi(1) = 7^4 \cdot 2^8 \cdot 3 \cdot 19$. From Frame's theorem

$$\frac{7^2 \cdot 9 \cdot 43 K_1 K_2}{2^5 \cdot 7 \cdot 19 \cdot 2 \cdot 3^2 \cdot 19 \cdot 43} = \frac{7 K_1 K_2}{2^6 \cdot 19^2} = (\text{integer})^2.$$

We obtain $7 \cdot 2^6 \cdot 19^2 | K_1 K_2$. We calculate $K_1 + K_2 = \varphi(1) - 1 = 2 \cdot 19 \cdot 499$ and $2^2 \nmid K_1 + K_2$. We can assume $2 | K_1$, $4 \nmid K_1$, $2^5 | K_2$, and $19 | K_i$ $(i = 1, 2)$. Hence the possibilities of $(1/2 \cdot 19) K_1$ and $(1/2 \cdot 19)(\varphi(1) - 1 - K_1) = (1/2 \cdot 19) K_2$ are in

$\frac{1}{2 \cdot 19} K_1$	1	7	7^2	7^3	3	$3 \cdot 7$	$3 \cdot 7^2$
$\frac{1}{2 \cdot 19} K_2$	498	492	450	156	496	478	352.

Since 498, 492, 450, 156, 478 are not divisible by 2^4 and 496, 352 cannot divide $|K|$, these are impossible.

LEMMA 4.9. *There exists no permutation representation φ^* of G^* with $G_2(3) \trianglelefteq G^*$ such that $\varphi(1) = 351$, $\chi(1) = 168$, $\xi(1) = 91$ and $k = 2$.*

Proof. Suppose φ^* is a counterexample. Then $\varphi = 1 + \chi + \xi_1 + \xi_2$, where ξ_1 and ξ_2 are conjugate to each other under G^* and $\xi_1(1) = \xi_2(1) = 91$. φ^* is of rank 3 and $\varphi = \varphi^*|_{G_2(3)}$ is of rank at least 4, so $G^* \neq G_2(3)$. Thus $G^* = \text{Aut } G_2(3)$.

From [10, (9.12)], ξ_1 and ξ_2 are not equivalent. We find from [5] that there is unique representation pair of this kind, namely ξ_1 and ξ_2 are θ_3 and θ_4, and that χ is just θ_2, where θ_2, θ_3, and θ_4 are the notation in [5]. Since $\varphi = 1 + \chi + \xi_1 + \xi_2$ is a permutation representation of $G_2(3)$, for any $g \in G_2(3)$ the value $\varphi(g)$ is a nonnegative integer. But from [5], there exist elements $g \in G_2(3)$ such that $1 + \theta_2 + \theta_3 + \theta_4$ is negative. This is a contradiction.

Now we can prove the main theorem. From the conclusion of Section 1 and the assertions of 4.1–4.9, if a faithful permutation representation φ^* of G^* is of rank 3, then

(1) $G_2(3) \trianglelefteq G^*$,

(2) $\varphi = 1 + \chi + \xi$, where $\varphi(1) = 351$, $\chi(1) = 168$ and $\xi(1) = 182$.

Suppose $G_2(3) \neq G^*$, then $G^* = \operatorname{Aut} G_2(3) = \langle G_2(3), \eta \rangle$, where η is the graph automorphism of $G_2(3)$. In [5], we find that χ is just θ_2 and ξ is χ_5 or χ_7. By looking at their character formulas, for $q = 3$, $1 + \theta_2 + \chi_5$ and $1 + \theta_2 + \chi_7$ exchange with each other under η. But $\varphi = \varphi^*|_{G_2(3)}$ and $G_2(3) \lhd G^*$, φ is self-conjugate under G^*. This is a contradiction. Thus $G_2(3) = G^*$ and the proof of the main theorem is complete.

5. Examples of the Rank 3 Permutation Representation

We shall follow the notation used in [2, 5, 6]. A group M is defined in [5, Sect. 4]: M is a subgroup of

$$\Gamma(w'_a) = \{ g \in \bar{G} \mid w_a'^{-1} g w'_a = g^{(q)} \},$$

and is isomorphic to $SU(3, q^2)$. For every element $g \in M$, $(g^{(q)})^{(q)} = g$. And from [13] we have that the action of $(w'_a)^2$ on $\{X_{\pm b}, X_{\pm(3a+b)}, X_{\pm(3a+2b)}\}$ is the identity. So w'_a normalizes M. Set

$$M_1 = \langle w'_a, M \rangle,$$

then M_1 is a group and $|M_1 : M| = 2$. It is obvious that $w'_a \in \Gamma(w'_a)$. Thus $M_1 \subseteq \Gamma(w'_a)$. From [2], $\Gamma(w'_a) \cong G_2(q)$.

Now $q = 3$; $|M_1| = 2 \cdot |SU(3, 3^2)| = 2 \cdot 3^3(3^3 + 1)(3^2 - 1) = 2^6 \cdot 3^3 \cdot 7$. We imbed M_1 into $G_2(3)$. Then the degree of $1_{M_1}^{G_2}$ is just equal to 351.

We will prove that $1_{M1}^{G_2(3)}$ is of rank 3. At first from [15, (3.4)], $SU(3, 3^2)$ has only one outer automorphism, i.e., the field automorphism. The action of w'_a on $M \cong SU(3, 3^2)$ is just the field automorphism of $SU(3, 3^2)$. It is known that $G_2(2)$ is equivalent to the automorphism group of $SU(3, 3^2)$. Thus $G_2(2) \cong M_1$.

From [5, Tables VI–1, VII–1, VII–2; 6, Table 1] we can determine the conjugacy classes of M_1 under $G_2(3)$.

notation of classes of $G_2(3)$	number of elements in the class
A_1	1
A_2	$2^3 \cdot 7$
A_{42}	$2^5 \cdot 3 \cdot 7$
B_1	$2^2 \cdot 3^2 \cdot 7 + 3^2 \cdot 7$
B_2	$2^3 \cdot 3^2 \cdot 7$
B_5	$2^5 \cdot 3^2 \cdot 7$
D_{11}	$2^2 \cdot 3^2 \cdot 7 + 2 \cdot 3^3 \cdot 7$
D_{12}	$2^4 \cdot 3^3 \cdot 7$
D_{21}	$2 \cdot 3^3 \cdot 7$
E_2	$2^3 \cdot 3^3 \cdot 7$
E_3	$2^3 \cdot 3^3 \cdot 7$
E_6	$2^6 \cdot 3^3.$

From [5, Table VII–2, characters of $G_2(3)$] and

$$(1_{M_1}^{G_2(3)}, \chi) = (1, \chi)_{M_1} = \frac{1}{|M_1|} \sum_{g \in M_1} \chi(g),$$

we obtain

$$(1_{M_1}^{G_2(3)}, \theta_2) = (1_{M_1}^{G_2(3)}, \chi_7) = 1.$$

Since $\theta_2(1) = 168$ and $\chi_7(1) = 182$, we have

$$1_{M_1}^{G_2(3)} = 1 + \theta_2 + \chi_7(1).$$

Hence $1_{M_1}^{G_2(3)}$ is of rank 3. Set

$$M_2 = (M_1)^\eta,$$

where η is the graph automorphism of $G_2(3)$. From [5, Table VII–2], χ_7 and χ_5 are conjugate to each other under η (only for $q = 3$) and θ_2 is self-conjugate. Thus

$$1_{M_2}^{G_2(3)} = 1 + \theta_2 + \chi_5$$

is another inequivalent permutation representation of rank 3 of $G_2(3)$.

We claim that $1_{M_1}^{G_2(3)}$ and $1_{M_2}^{G_2(3)}$ are primitive, i.e., M_i $(i = 1, 2)$ are maximal subgroups of $G_2(3)$. Suppose $M_i \subsetneqq K \subsetneqq G_2(3)$ $(i = 1$ or $2)$. Since $1_{M_i}^{G_2(3)}$ is of rank 3, we assume the decomposition of $M_i - M_i$ double cosets of $G_2(3)$ is

$$G = M_i \cup M_i a M_i \cup M_i b M_i.$$

From $M_i \subset K$, we have $G = K \cup KaK \cup KbK$. And from $M_i \subsetneqq K$, we get $K \cap M_i a M_i \neq \varnothing$ or $K \cap M_i b M_i \neq \varnothing$. Thus $K \cap KaK \neq \varnothing$ or $K \cap KbK \neq \varnothing$ and therefore $K = KaK$ or $K = KbK$. And since $K \neq G_2(3)$, $1_K^{G_2(3)}$ is of rank 2, i.e., $1_K^{G_2(3)}$ is doubly transitive. But from [4], $G_2(3)$ has no doubly transitive permutation representation. This is a contradiction. Hence M_i $(i = 1, 2)$ are maximal and $1_{M_i}^{G_2(3)}$ $(i = 1, 2)$ are primitive.

$1_{M_i}^{G_2(3)}$ $(i = 1, 2)$ are two examples of primitive permutation representation of rank 3 of $G_2(3)$.

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REFERENCES

1. B. CHANG AND R. REE, The characters of $G_2(q)$, Symposia Mathematica XIII (1974), 395–413.
2. B. CHANG, The conjugacy classes of Chevalley groups of type (G_2), *J. Algebra* **9** (1968), 190–211.
3. C. W. CURTIS AND I. REINER, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.
4. C. W. CURTIS, W. M. KANTOR, AND G. M. SEITZ, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* **218** (1976), 1–59.
5. H. ENOMOTO, The characters of $G_2(q)$, $q = 3^f$, *Japan. J. Math.* (*N.S.*) **2** (1976), 191–248.
6. H. ENOMOTO, The conjugacy class of Chevalley groups of type (G_2) over finite fields of characteristic 2 or 3, *J. Fac. Sci. Univ. Tokyo Sect. IA*, Math. **XVI** (1970), 497–512.
7. W. FEIT, On finite linear groups, *J. Algebra* **5** (1967), 378–400.
8. W. FEIT AND J. G. THOMPSON, Groups which have a faithful representation of degree less than $(p-1)/2$, *Pacific J. Math.* **11** (1961), 1257–1262.
9. W. FEIT, Groups which have a faithful representation of degree less than $p-1$, *Trans. Amer. Math. Soc.* **112** (1964), 287–303.
10. W. FEIT, "Characters of Finite Groups," Benjamin, New York, 1967.
11. J. S. FRAME, The degree of the irreducible components of simply transitive groups, *Duke Math. J.* **3** (1937), 8–17.
12. D. S. PASSMAN, Solvable half-transitive automorphism groups, *J. Algebra* **6** (1967), 285–304.
13. R. REE, A family of simple groups associated with the simple Lie algebra of type G_2, *Amer. J. Math.* **83** (1961), 432–462.
14. G. M. SEITZ, Flag-transitive subgroup of Chevalley groups, *Ann. of Math.* **97** (1973), 27–56.
15. R. STEINBERG, Automorphisms of finite linear groups, *Canad. J. Math.* **12** (1960), 606–615.
16. R. STEINBERG, Lecture Notes on Chevalley groups, Yale Univ., New Haven, Conn. 1967.